Some Results on a Paint Shop Problem for Words

Th. Epping, W. Hochstättler, P. Oertel [1]

Center for Applied Computer Science, University of Cologne

1 Introduction

Motivated by an application in the automobile industry, we present results and conjectures on a new combinatorial problem.

We are given a word $w = (w_1, \ldots, w_n)$ with letters w_i of an alphabet B, and a color vector $f = (f_1, \ldots, f_n)$ with colors f_i of a color set F. Each f_i denotes the color of w_i. Whenever $f_i \neq f_{i+1}$, we say that we have a *color change* in f.

Problem 1 *Paint Shop Problem for Words (PPW)*

Given a finite alphabet B, a word $w = (w_1, \ldots, w_n) \in B^*$, a set F of colors and a coloring $f = (f_1, \ldots, f_n)$ of w, find a permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that $w_{\sigma(i)} = w_i$ for $i = 1, \ldots, n$ and the number of all color changes within $\sigma(f) = (f_{\sigma(1)}, \ldots, f_{\sigma(n)})$ is minimized.

Given an instance $(w; f)$ of PPW, we denote the number of its color changes by $\gamma(w)$ and the optimal (minimal) number of color changes by $\gamma^*(w)$. Note, that the initial coloring of a word determines the reservoir of letters available in each color. Thus, we can deal with these reservoirs instead of a color vector. We denote the *reservoir* of letter i in color j by $V(i, j)$.

Considering the letters w_i as car bodies, Problem 1 refers to the problem of coloring a given car body sequence in a paint shop. As each color change gives rise to substantial cost and pollution, the minimization of color changes is aspired by the automobile industry for a long time (see [1] and references therein).

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2 Complexity Results

Even restricted versions of Problem 1 are $\mathcal{NP}$-complete. We restrict to instances of bounded size of F resp. B and show by reduction from 3SAT resp. pseudo-polynomial reduction from 3-PARTITION that even in these cases the problem remains $\mathcal{NP}$-complete.

Theorem 2 *PPW is $\mathcal{NP}$-complete for $|F| = 2$.*

Theorem 3 *PPW is $\mathcal{NP}$-complete for $|B| = 2$.*

Each instance $(w; f)$ of PPW can be solved by a dynamic program. We only have to pass through w (letter by letter from the left to the right) and record each feasible coloring up to the current position in a different state.

Theorem 4 *An instance of PPW with letters of an alphabet B and colors of a color set F can be solved with a space and time complexity of $O(|F|n^{|F||B|})$.*

Details on the results of this section can be found in [2].

3 k-regular Instances

Given a fixed integer $k \geq 1$, we call an instance $(w; f)$ of PPW k-regular, if $V(i, j) = k = \frac{n}{|B||F|}$ holds for all letters i and colors j. We first restrict to the case of k-regular instances of bounded size of B and give an upper bound for the value $\gamma^*(w)$, starting with a simple lemma.

Lemma 5 *Suppose we are given a k-regular instance of PPW with $|B| = |F| = 2$. Then $\gamma^*(w) \leq 2$ holds.*

We use Lemma 5 to prove Theorem 6 by induction.

Theorem 6 *Suppose we are given a k-regular instance of PPW with $|B| = 2$. Then $\gamma^*(w) \leq 2(|F| - 1)$ holds.*

Indeed, besides the dynamic program mentioned in Theorem 4 we know no efficient way to compute an optimal coloring (even for k-regular instances). When dealing with instances of bounded size of F instead of B, we can not even show an upper bound for an optimal coloring (like in Theorem 6). Thus, we only present a conjecture for this case.

Conjecture 7 *Suppose we are given a k-regular instance of PPW with $|F| = 2$. Then $\gamma^*(w) \leq |B|$ holds.*

Note, that Conjecture 7 is correct for $k = 1$. Combining Theorem 6 and Conjecture 7 results in Conjecture 8.

Conjecture 8 *Suppose we are given a k-regular instance of PPW. Then $\gamma^*(w) \leq |B|(|F| - 1)$ holds.*

The following example proves that the bound given in Conjecture 8 is tight if the conjecture is correct.

Example 9 *Suppose we are given a k-regular instance of PPW with a color set F and an alphabet $B = \{b_1, \ldots, b_{|B|}\}$ of the form*

$$w = (\underbrace{b_1 \ldots b_1}_{k|F|}\underbrace{b_2 \ldots b_2}_{k|F|} \ldots \underbrace{b_{|B|} \ldots b_{|B|}}_{k|F|}).$$

Then $\gamma^*(w) = |B|(|F| - 1)$ holds.

Finally, we take a look at 1-regular instances $(w; f)$ with $|F| = 2$. One might expect that the natural greedy approach (when coloring w from the left to the right, keep the actual color as long as possible) produces good results. Example 10 shows that in general this is not the case.

Example 10 *Suppose we are given a 1-regular instance of PPW with $|F| = 2$ and $B = \{b_1, \ldots, b_{|B|}\}$ (with $|B|$ even) of the form*

$$w = (\underbrace{b_1 \ldots b_{|B|/2}}\underbrace{b_{|B|/2} \ldots b_{|B|}}\, b_{|B|}\, \underbrace{b_1 b_{|B|/2+1}}\underbrace{b_2 b_{|B|/2+2}} \ldots \underbrace{b_{|B|/2-1} b_{|B|-1}}).$$

Tbe greedy algorithm defined above colors the word w with $|B| = \frac{n}{2} = O(n)$ color changes, while the minimal number of color changes is always $\gamma^*(w) = 3$.

We end with an open problem.

Problem 11 *Given a 1-regular instance $(w; f)$ of PPW with $|F| = 2$, compute or approximate the optimal value $\gamma^*(w)$.*

References

[1] S. Spieckermann and S. Voß: *Paint Shop Simulation in the Automotive Industry.* Preprint, 2000

[2] Th. Epping, W. Hochstättler, P. Oertel: *A Paint Shop Problem for Words.* Technical Report No. 00.395, Center for Parallel Computing, University of Cologne, 2000